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Weak solutions for equations defined by accretive operators II: relaxation limits

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Abstract

In the first part of this paper we define solutions for certain nonlinear equations defined by accretive operators, "dissipative solution". This kind of solution is equivalent to the viscosity solutions for Hamilton–Jacobi equations and to the entropy solutions for conservation laws.

In this paper we use dissipative solutions to obtain several relaxation limits for systems of semilinear transport equations and quasilinear conservation laws. These converge to diffusion second-order equations and in one case to a single conservation law. The relaxation limit is obtained using a version of the perturbed test function method to pass to the limit. This guarantees existence for the considered equations.

Keywords: Weak solutions; Contractive equations; Accretive operators; Viscosity solutions; Entropy solutions; Dissipative solutions

1. Introduction

In the first part of this paper [10] we introduced a notion of solution of equations defined by accretive operators. This was shown to be parallel to the notion of *viscosity solutions* ([1,2]) for Hamilton–Jacobi equations and *entropy solutions* ([7]) of conservation laws. Here we use this to obtain certain relaxation limits for hyperbolic systems collapsing to parabolic equations.

The main idea to obtain these limits is the *perturbed test function method* which Evans used in [3,4] and in the context of the Hamilton–Jacobi equation. It is interesting to see that this method also applies to conservation laws, using the notion

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of dissipative solution. In fact it is this form of expressing the solution which makes the computations feasible. It is not clear how to do this using Kružkov's formulation directly.

One of the limits we present was obtained by Kurtz in [8], where a system of two equation in one space variable converges to a nonlinear parabolic equation, but using more difficult techniques. Here we obtain similar results for more general systems in $\mathbb{R}^n$.

Let us recall the relevant definitions and basic results from [10]. Given an equation written as

$$Au = f,$$

where $A : D(A) \to 2^X$ is a (possibly) multivalued *accretive* operator defined on a subset of some Banach space X, we will say that u is a *dissipative* solution of this equation if it satisfies

$$[u - \phi, f - A\phi]_+ \geqslant 0$$

for every ϕ in some nice class of functions. Here $[\cdot,\cdot]_+$ is the *Kato bracket*, defined by

$$[u, v]_+ := \lim_{\lambda \to 0^+} \frac{||u + \lambda v|| - ||u||}{\lambda}.$$

This definition is consistent when A is accretive, since for this type of operator we always have

$$[u - v, Au - Av]_+ \geqslant 0$$

for every $u, v \in D(A)$.

In $L^1(d\mu)$ the Kato bracket is given by

$$[u, v]_+ = \int_{\{u \neq 0\}} \operatorname{sgn}(u) v \, d\mu + \int_{\{u = 0\}} |v| \, d\mu.$$

This motivates the following extension of the operator $\operatorname{div} \mathbf{F}(\cdot)$: If $u \in L^1(\mathbb{R}^n)$ we say that $u \in D(A)$ and $v \in Au$ provided $v \in L^1(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} \operatorname{sgn}(u - \phi)(v - \operatorname{div} \mathbf{F}(\phi)) \, dx \geqslant 0,$$

for every $\phi \in C^1(\mathbb{R}^n)$ such that $\phi(x) \equiv$ constant for x large enough. It is possible to show that this operator is accretive and that the dissipative solutions of $Au = f$ correspond precisely to the Kružkov solutions [10].

Here we will deal with several relaxation limits using the perturbed test function method. This method works well with our formulation of weak solution. That is not the case with Kružkov's definition.

The first system we look at was studied by Katsoulakis and Tzavaras in [5]. For $w^\varepsilon : \mathbb{R}^n \to \mathbb{R}$, $\mathbf{z}^\varepsilon = (z_1, \dots, z_n^\varepsilon) : \mathbb{R}^n \to \mathbb{R}^n$,

$$\begin{cases} w_t^\varepsilon + \sum_{i=1}^n \omega_i v_i w_{x_i}^\varepsilon = -\frac{1}{\varepsilon} \sum_{i=1}^n (h_i(w^\varepsilon) - z_i^\varepsilon), \\ z_{i,t}^\varepsilon - v_i z_{i,x_i}^\varepsilon = -\frac{1}{\varepsilon} (h_i(w^\varepsilon) - z_i^\varepsilon), \quad i = 1, \dots, n, \end{cases}$$

where $\omega_1, \dots, \omega_n$ and $v_1, \dots, v_n$ are constants. This system is a discrete velocity kinetic equation for the state variables $(w, \mathbf{z})$.

Katsoulakis and Tzavaras prove that this system is a contraction in L^1 and in the limit as $\varepsilon \to 0$ it becomes (equivalent to) a conservation law. We prove the same using the new formulation of solution (which we know is equivalent).

We then look at the same system but with a different scaling:

$$\begin{cases} w_t^\varepsilon + \frac{1}{\varepsilon} \sum_{i=1}^n w_{x_i}^\varepsilon = -\frac{1}{\varepsilon^2} \sum_{i=1}^n (h_i(w^\varepsilon) - h_i(z_i^\varepsilon)) \\ z_{i,t}^\varepsilon - \frac{1}{\varepsilon} z_{i,x_i}^\varepsilon = \frac{1}{\varepsilon^2} (h_i(w^\varepsilon) - h_i(z_i^\varepsilon)), \quad i = 1, \dots, n. \end{cases}$$

This corresponds to looking at the long time behavior of the solutions of the original system. In this case we prove the convergence to a quasilinear parabolic equation:

$$w_t - \frac{1}{n+1} \sum_{i=1}^n \left(\frac{w_{x_i}}{h_i'(w)} \right)_{x_i} = 0.$$

Note that there is some analogy between these two convergence results and the laws of large numbers and the Central Limit Theorem: In the first of these systems we add the equations in the system to get an averaging effect for the limit. In the second system a more subtle scaling reveals more structure for the limiting function.

We also look at a quasilinear generalization of Kurtz's example converging to the nonlinear diffusion PDE

$$u_t - \frac{1}{2} \left(\frac{(f'(u))^2}{\beta'(u)} u_x \right)_x = 0.$$

Finally we take a more general system

$$\begin{cases} w_t^\varepsilon + \frac{1}{\varepsilon} \sum_{i=1}^n \sum_{j=1}^n f_{ij}(w^\varepsilon)_{x_j} = -\frac{1}{\varepsilon^2} \sum_{i=1}^n (w^\varepsilon - z_i^\varepsilon) \\ z_{i,t}^\varepsilon - \frac{1}{\varepsilon} \sum_{j=1}^n f_{ij}(z_i^\varepsilon)_{x_j} = \frac{1}{\varepsilon^2} (w^\varepsilon - z_i^\varepsilon), \quad i = 1, \dots, n, \end{cases}$$

and show that in the limit as $\varepsilon \to 0$ we find the parabolic equation

$$u_t - \frac{1}{n+1} \sum_{j=1}^n \sum_{k=1}^n (a_{jk}(u)\, u_{x_k})_{x_j} = 0,$$

where a_{ij}, is a nonnegative definite symmetric matrix: $((a_{jk})) = \sigma^T \sigma$, with $\sigma = ((f_{ij}'))$.

We show the limiting equation defines an accretive operator and then take the limit in the above system, using the same method of the perturbed test function. From earlier results in the section we see that we can write the matrix a_{ij} in this form while getting an accretive operator for this system, so our computation is justified.

2. Relaxation effects for conservation laws

We consider first the following system for w^ε and $\mathbf{z}^\varepsilon = (z_1^\varepsilon, \dots, z_n^\varepsilon)$:

$$\begin{cases} w_t^\varepsilon + \sum_{i=1}^n \omega_i v_i w_{x_i}^\varepsilon = -\frac{1}{\varepsilon} \sum_{i=1}^n (h_i(w^\varepsilon) - z_i^\varepsilon) \\ z_{i,t}^\varepsilon - v_i z_{i,x_i}^\varepsilon = -\frac{1}{\varepsilon}(h_i(w^\varepsilon) - z_i^\varepsilon), \quad i = 1, \dots, n, \end{cases} \tag{1}$$

where $\omega_1, \dots, \omega_n$ and $v_1, \dots, v_n$ are constants (eventually to be chosen appropriately).

This system can be interpreted as a discrete velocity kinetic equation where the state vector $(w, \mathbf{z})$ lives on the "physical" space $\mathbb{R}^n \times [0, \infty)$. The quantity w is convected with the velocity $(\omega_1 v_1, \dots, \omega_n v_n)$, and z_i with $-v_i e_i$. The functions h_i describe the interactions of w with z_i. Tzavaras and Katsoulakis interpret this system as a mesoscopic scaling limit of a stochastic interacting particle system (see [6]).

Katsoulakis and Tzavaras [5] then show that this PDE generates a contraction in $L^1([0, \infty) \times \mathbb{R}^n; \mathbb{R}^{n+1})$, uniformly in ε, provided $h_i' > 0$, $h_i(0) = 0$ and $|h_i(x)| \to \infty$ as $|x| \to \infty$. This implies that $w_t^\varepsilon, w_{x_i}^\varepsilon, \mathbf{z}_t^\varepsilon$ and $\mathbf{z}_{x_i}^\varepsilon$, for $1 \leqslant i \leqslant n$, are bounded in L^1, which in turn implies that w^ε and $\mathbf{z}^\varepsilon$ are bounded in BV, the space of functions of bounded variation. This gives us compactness in L^1, so we can extract a subsequence w^{ε_j}, $\mathbf{z}^{\varepsilon_j}$ converging in L^1, and then extract a further subsequence converging almost everywhere.

Therefore, when we let $\varepsilon_j \to 0$, we have $w^{\varepsilon_j} \to w$ and $z_i^{\varepsilon_j} \to h_i(w)$, and w solves

$$\left(w + \sum_{i=1}^n h_i(w) \right)_t + \operatorname{div}(v_1(\omega_1 w - h_1(w)), \dots, v_n(\omega_n w - h_n(w))) = 0$$

in the entropy sense, that is, in the sense of the Kružkov definition. This can be interpreted as a model for a chemical reaction between w and each of the components in z.

We can convert this into the more familiar form of a conservation law by defining $r(s) := (1 + \Omega)^{-1}(s + \sum_{i=1}^n v_i^{-1} \mathbf{F}_i(s))$, where $\Omega = \sum \omega_i$, $\mathbf{F}_i(s) = \omega_i v_i s - v_i h_i(s)$. With this construction, $r^{-1}(t) = t + \sum h_i(t)$. Defining $u = r^{-1}(w)$, we get an entropy solution of the conservation law

$$u_t + \operatorname{div}(\mathbf{F}(u)) = 0. \tag{CL}$$

Of course, it is interesting as well to start with this conservation law and then find system (1) which leads to this equation. This is possible in general, under a reasonable assumption on $\mathbf{F}$. If we can find ω_i, v_i, such that

$$1+\sum_{i=1}^{n}\frac{1}{v_i}\mathbf{F}_i'>0 \quad \text{and} \quad (1+\Omega)\frac{1}{v_j}\mathbf{F}_j'<\left(1+\sum_{i=1}^{n}\frac{1}{v_i}\mathbf{F}_i'\right)\omega_j,$$

then it is possible to find functions r and h_i, $i=1,\dots,n$, such that the above relations between $\mathbf{F}$, h_i and r hold, h_i satisfy the above conditions. The limit function $u=r^{-1}(w)$ is the entropy solution of the given conservation law.

We want to illustrate that this limit is also a dissipative solution of (CL), that is, for every smooth function ζ,

$$0\leqslant \iint_{\{u\neq\zeta\}} \operatorname{sgn}(u-\zeta)(-\zeta_t-\operatorname{div}(\mathbf{F}(\zeta)))\,dx\,dt. \tag{2}$$

Note that we already know this by the equivalence of the two notions stated above. However this serves to illustrate the technique we will apply later and gives another proof of the result of Katsoulakis and Tzavaras.

Theorem 2.1. *There exist subsequences $w^\varepsilon, z_i^\varepsilon \in L^1([0,\infty)\times\mathbb{R}^n)$ $(1\leqslant i\leqslant n)$ converging to $w, h_i(w)$, respectively, such that, with r defined as above, the function $u=r^{-1}(w)$ is a dissipative solution of* (2).

Proof. 1. Given ζ a test function as above, let $\phi := r(\zeta)$ and $\psi_i := h_i(\phi)$, and plug these functions into the "bracket" definition of solution to the system (1):

$$\begin{aligned}0\leqslant \iint \operatorname{sgn}(w^\varepsilon-\phi)&\left[-\phi_t-\sum_{i=1}^{n}\omega_i v_i\phi_{x_i}-\frac{1}{\varepsilon}\sum_{i=1}^{n}(h_i(\phi)-\psi_i)\right]\\ &+\sum_{i=1}^{n}\operatorname{sgn}(z_i^\varepsilon-\psi_i)\left[-\psi_{i,t}+v_i\psi_{i,x_i}-\frac{1}{\varepsilon}(h_i(\phi)-\psi_i)\right]dx\,dt.\end{aligned} \tag{3}$$

The terms with $1/\varepsilon$ are zero.

2. Assume for now that $\operatorname{sgn}(w^\varepsilon-\phi)\to\operatorname{sgn}(w-\phi)$, $\operatorname{sgn}(z_i^\varepsilon-\psi_i)\to\operatorname{sgn}(z_i-\psi_i)$ a.e. and let $\varepsilon\to 0$. Noting that $h_i'>0$, $\operatorname{sgn}(h_i(w)-h_i(\phi))=\operatorname{sgn}(w-\phi)$ and we have

$$0\leqslant \iint \operatorname{sgn}(w-\phi)\left[-\phi_t-\sum_{i=1}^{n}h_i(\phi)_t-\sum_{i=1}^{n}(\omega_i v_i\phi-v_i h_i(\phi))_{x_i}\right]dx\,dt. \tag{4}$$

Now, since $r^{-1}(s)=s+\sum h_i(s)$ and $\mathbf{F}_i(s)=\omega_i v_i s-v_i h_i(s)$, replacing $w=r(u)$, $\phi=r(\zeta)$, we get

$$0\leqslant \iint \operatorname{sgn}(r(u)-r(\zeta))\left[-r^{-1}(\phi)_t-\sum_{i=1}^{n}\mathbf{F}_i(r^{-1}(\phi))_{x_i}\right]dx\,dt. \tag{5}$$

Finally we notice that $r^{-1}(\phi) = \zeta$ and since r is strictly increasing, $\operatorname{sgn}(r(u) - r(\zeta)) = \operatorname{sgn}(u - \zeta)$, thus (2) in fact holds.

Above we assumed that $\operatorname{sgn}(w^\varepsilon - \phi) \to \operatorname{sgn}(w - \phi)$, $\operatorname{sgn}(z_i^\varepsilon - \psi_i) \to \operatorname{sgn}(z_i - \psi_i)$ a.e. If we pass to a subsequence of $(w^\varepsilon, z_i^\varepsilon)$ which converges for a.e. $x \in \mathbb{R}^n$ we also have $\operatorname{sgn}(w^\varepsilon - \phi) \to \operatorname{sgn}(w - \phi)$, $\operatorname{sgn}(z_i^\varepsilon - \psi_i) \to \operatorname{sgn}(z_i - \psi_i)$ a.e., but only in the set $\{w \neq \phi, z_i \neq \psi_i\}$.

We devote the rest of this section to justifying the claim.

3. By the above observation, it is clear that (5) holds if $|\{w \neq \phi\}| = 0$. Now let ϕ be smooth, but where $|\{w \neq \phi\}|$ does not necessarily vanish.

Let $\psi \in C_c^\infty$, ψ nonnegative and positive in a large ball B containing the support of ϕ. Look at the test functions $\phi_\delta \coloneqq \phi + \delta\psi$.

We will show that there exists a sequence $\delta_j \to 0$, such that $|\{w = \phi_{\delta_j}\}| = 0$. Let $A_\delta = \{w = \phi_\delta\} \cap \operatorname{supp}(\phi)$. Then for each $\delta, \delta^* > 0$, whenever $x \in A_\delta \cap A_{\delta^*}$ we have $w(x) = \phi(x) + \delta\psi(x)$ and $w(x) = \phi(x) + \delta^*\psi(x)$, which implies $\varepsilon = \delta$. Therefore the sets A_ε are pairwise disjoint and this means that at most countably many of these sets have positive measure. So we can easily pick a sequence as we claimed.

4. Now apply (5) to ϕ_{δ_j}, written as

$$\begin{aligned} 0 \leqslant & \iint_{\{w \neq \phi\}} \operatorname{sgn}(w - \phi_{\delta_j})(\phi_t + \delta_j\psi_t + \operatorname{div}\mathbf{F}(\phi + \delta_j\psi))\,dx\,dt \\ & - \iint_{\{w = \phi\}} (\phi_t + \delta_j\psi_t + \operatorname{div}\mathbf{F}(\phi + \delta_j\psi))\,dx\,dt. \end{aligned}$$

Noting that $\operatorname{sgn}(w - \phi_{\delta_j}) \to \operatorname{sgn}(w - \phi)$, a.e. on $\{w \neq \phi\}$, we find that

$$0 \leqslant \iint_{\{w \neq \phi\}} \operatorname{sgn}(w - \phi)(\phi_t + \operatorname{div}\mathbf{F}(\phi))\,dx\,dt - \iint_{\{w = \phi\}} (\phi_t + \operatorname{div}\mathbf{F}(\phi))\,dx\,dt.$$

If we use as test functions $\phi - \delta\psi$ we will find instead that

$$0 \leqslant \iint_{\{w \neq \phi\}} \operatorname{sgn}(w - \phi)(\phi_t + \operatorname{div}\mathbf{F}(\phi))\,dx\,dt + \iint_{\{w = \phi\}} (\phi_t + \operatorname{div}\mathbf{F}(\phi))\,dx\,dt.$$

Thus, adding these two inequalities we get (2), as claimed. $\square$

3. A nonlinear diffusion limit

Let us now consider a different scaling for system (1):

$$\begin{cases} w_t^\varepsilon + \frac{1}{\varepsilon}\sum_{i=1}^n w_{x_i}^\varepsilon = -\frac{1}{\varepsilon^2}\sum_{i=1}^n (h_i(w^\varepsilon) - h_i(z_i^\varepsilon)) \\ z_{i,t}^\varepsilon - \frac{1}{\varepsilon} z_{i,x_i}^\varepsilon = \frac{1}{\varepsilon^2}(h_i(w^\varepsilon) - h_i(z_i^\varepsilon)), \quad i = 1, \ldots, n. \end{cases} \tag{6}$$

This is a special case of (1) with a scaling like $(x, t/\varepsilon)$. So we can interpret this limit as the long time behavior of the interactions in (1). We can also draw a loose analogy between (6) and (1), and the difference between the Law of Large Numbers and the Central Limit Theorem. Indeed, with the right averaging we get more information about the limit, namely that it is a nonlinear diffusion equation. See Pinsky [9] for more discussion of this point when the relevant PDE are linear.

This system is a contraction and $w^\varepsilon \to w$, $z_i^\varepsilon \to w$ for suitable subsequences. The question now is what equation should w satisfy?

As a motivation, we do the following formal calculation: Take $v^\varepsilon = w^\varepsilon + \sum_j z_j^\varepsilon$. On one hand, we expect v^ε to converge to $(n+1)w$, and "thus" $v_t^\varepsilon \to (n+1)w_t$. On the other hand,

$$\begin{aligned} v_t^\varepsilon &= -\frac{1}{\varepsilon}\sum_{i=1}^n (w^\varepsilon - z_i^\varepsilon)_{x_i} = -\frac{1}{\varepsilon}\sum_{i=1}^n \left[(w^\varepsilon - z_i^\varepsilon)\varepsilon\frac{\varepsilon z_{i,t}^\varepsilon - z_{i,x_i}^\varepsilon}{h_i(w^\varepsilon) - h_i(z_i^\varepsilon)}\right]_{x_i} \\ &= \sum_{i=1}^n \left[\frac{w^\varepsilon - z_i^\varepsilon}{h_i(w^\varepsilon) - h_i(z_i^\varepsilon)}(z_i^\varepsilon - \varepsilon z_{i,t})\right]_{x_i}, \end{aligned}$$

where we used (6) for z_i on the second line above.

If we could carry the limit as $\varepsilon \to 0$ inside the derivative with respect to x_i in the above formula, we should then have

$$w_t - \frac{1}{n+1}\sum_{i=1}^n \left(\frac{w_{x_i}}{h_i'(w)}\right)_{x_i} = 0. \tag{7}$$

Theorem 3.1. *There exist subsequences of $w^\varepsilon, z_i^\varepsilon \in L^1([0,\infty)\times\mathbb{R}^n)$ converging to w, where w is a dissipative solution of* (7).

Proof. 1. We know (6) has a classical solution which is also dissipative solution. Given ϕ smooth, take for smooth functions in the definition of dissipative solution of (6) the following functions:

$$\phi^\varepsilon := \phi, \quad \psi_i^\varepsilon := \phi + \varepsilon\psi_{i*}, \quad \text{where } \psi_{i*} = \frac{\phi_{x_i}}{h'(\phi)}.$$

We guess this by adapting the argument of Evans (see [3,4]) of the perturbed test function, only now we are working in L^1 rather than in $C(K)$. The purpose of the perturbation ψ_{i*} is to have cancellation of the "bad" terms, with a negative power of ε in front.

Thus we have

$$\begin{aligned} 0 \leqslant \iint & \operatorname{sgn}(w^\varepsilon - \phi)\left[-\phi_t - \frac{1}{\varepsilon}\sum_{i=1}^n \phi_{x_i} - \frac{1}{\varepsilon^2}\sum_{i=1}^n (h_i(\phi) - h_i(\psi_i^\varepsilon))\right] \\ &+ \sum_{i=1}^n \operatorname{sgn}(z_i^\varepsilon - \psi_i^\varepsilon)\left[-\psi_i^\varepsilon + \frac{1}{\varepsilon}\psi_{i,x_i}^\varepsilon + \frac{1}{\varepsilon^2}(h_i(\phi) - h_i(\psi_i^\varepsilon))\right] dx\, dt. \end{aligned} \tag{8}$$

Using now

$$f(x+h) = f(x) + hf'(x) + h^2 \int_0^1 (1-s)f''(x+sh)\,ds \tag{9}$$

for each h_i, we have

$$\begin{aligned} h_i(\phi) - h_i(\psi_i^\varepsilon) &= -\varepsilon\psi_{i*}h_i'(\phi) - \varepsilon^2\psi_{i*}^2 \int_0^1 (1-s)h_i''(\phi + s\,\varepsilon\,\psi_{i*})\,ds \\ &= -\varepsilon\phi_{x_i} + \varepsilon^2 C_i^\varepsilon, \end{aligned}$$

where we define $C_i^\varepsilon := -\psi_{i*}^2 \int_0^1 (1-s)h_i''(\phi + s\varepsilon\psi_{i*})\,ds$.

Now we put this back in (8) to find

$$\begin{aligned} 0 \leqslant \iint & \operatorname{sgn}(w^\varepsilon - \phi)\left[-\phi_t - \sum_{i=1}^n C_i^\varepsilon\right] \\ & + \sum_{i=1}^n \operatorname{sgn}(z_i^\varepsilon - \psi_i^\varepsilon)(-\phi_t - \varepsilon\psi_{i*,t} + \psi_{i*,x_i} + C_i^\varepsilon)\,dx\,dt. \end{aligned} \tag{10}$$

2. As we did in the proof of Theorem 3.1, assume that as $\varepsilon \to 0$, $\operatorname{sgn}(z_i^\varepsilon - \psi_i^\varepsilon) \to \operatorname{sgn}(w - \phi)$. Noting that $C_i^\varepsilon \to C_i = -\frac{1}{2}\psi_{i*}^2 h_i''(\phi)$, we have in the limit

$$\begin{aligned} 0 &\leqslant \iint \operatorname{sgn}(w-\phi)\left[-\phi_t + -\sum_{i=1}^n C_i + \sum_{i=1}^n \left(-\phi_t + \psi_{i*,x_i} + C_i\right)\right] dx\,dt \\ &\leqslant \iint \operatorname{sgn}(w-\phi)\left(-(n+1)\phi_t + \sum_{i=1}^n \psi_{i*,x_i}\right) dx\,dt. \end{aligned}$$

Since $\psi_{i*} = \phi_{x_i}/h'(\phi)$, it follows that

$$0 \leqslant \iint \operatorname{sgn}(w-\phi)\left[-(n+1)\phi_t + \sum_{i=1}^n \left(\frac{\phi_{x_i}}{h'(\phi)}\right)_{x_i}\right] dx\,dt. \tag{11}$$

3. To justify passing to the limit $\operatorname{sgn}(w - \phi_\varepsilon)$ we can proceed as in the proof of Theorem 3.1. Therefore the limit of (6) is in fact (7) in the dissipative sense. □

4. A model for a particle system

Consider now the following nonlinear relaxed system:

$$\begin{cases} u_t^\varepsilon + \frac{1}{\varepsilon} f(u^\varepsilon)_x = \frac{1}{\varepsilon^2}(\beta(u^\varepsilon) - \beta(v^\varepsilon)) \\ v_t^\varepsilon - \frac{1}{\varepsilon} f(v^\varepsilon)_x = \frac{1}{\varepsilon^2}(\beta(v^\varepsilon) - \beta(u^\varepsilon)). \end{cases} \tag{12}$$

A particular case of this system was studied by Kurtz [8]: with $f(z) = z$ and $\beta(z) = z^2$, but using much more difficult techniques. We can interpret this as system with two different kinds of particles which move on the line and interact according to the right-hand side of (12).

Adding both equations we get

$$(u^\varepsilon + v^\varepsilon)_t + \frac{1}{\varepsilon}(f(u^\varepsilon) - f(v^\varepsilon))_x = 0,$$

and using the first equation we find

$$\begin{aligned} \frac{1}{\varepsilon}(f(u^\varepsilon) - f(v^\varepsilon))_x &= \left[\frac{\varepsilon u_t^\varepsilon + f(u^\varepsilon)_x}{\beta(v^\varepsilon) - \beta(u^\varepsilon)} (f(u^\varepsilon) - f(v^\varepsilon)) \right]_x \\ &= - \left[(\varepsilon u_t^\varepsilon + f(u^\varepsilon)_x) \frac{u^\varepsilon - v^\varepsilon}{\beta(u^\varepsilon) - \beta(v^\varepsilon)} \frac{f(u^\varepsilon) - f(v^\varepsilon)}{u^\varepsilon - v^\varepsilon} \right]_x. \end{aligned}$$

As before, if we could take the limit inside the derivative with respect to x, and assuming that u^ε and v^ε have the same limit u, we would get

$$u_t - \frac{1}{2}\left[\frac{(f'(u))^2}{\beta'(u)} u_x \right]_x = 0. \tag{13}$$

Once again, (12) corresponds to a contraction in L^1, and therefore the limit, u, of some subsequence of u^ε and v^ε exists (and is in L^1).

Theorem 4.1. *The limit*, u, *of an appropriate subsequence* $u^\varepsilon, v^\varepsilon$, *is a dissipative solution of* (13).

Proof. 1. We need to show that for all smooth ϕ,

$$0 \leqslant \iint \operatorname{sgn}(u - \phi) \left[-\phi_t + \frac{1}{2}\left(\frac{(f'(\phi))^2}{\beta'(\phi)} \phi_x \right)_x \right] dx\, dt. \tag{14}$$

As before, the (classical) solution of (12) is also a dissipative solution, so we can test it against a pair $(\phi^\varepsilon, \psi^\varepsilon)$. Given ϕ, let

$$\phi^\varepsilon = \phi - \varepsilon p, \quad \psi^\varepsilon = \phi + \varepsilon p, \quad \text{with perturbation } p = \frac{f'(\phi)}{2\beta'(\phi)} \phi_x.$$

We choose this perturbation to exactly cancel the singular terms once we plug it into $f(\phi^\varepsilon)$. We know that

$$0 \leqslant \iint \operatorname{sgn}(u^\varepsilon - \phi^\varepsilon)\left(-\phi^\varepsilon_t - \frac{1}{\varepsilon} f(\phi^\varepsilon)_x + \frac{1}{\varepsilon^2}(\beta(\psi^\varepsilon) - \beta(\phi^\varepsilon))\right) + \operatorname{sgn}(v^\varepsilon - \psi^\varepsilon)\left(-\psi^\varepsilon_t + \frac{1}{\varepsilon} f(\psi^\varepsilon)_x + \frac{1}{\varepsilon^2}(\beta(\phi^\varepsilon) - \beta(\psi^\varepsilon))\right) dx\, dt.$$

2. From (9) and the definitions of ϕ^ε and ψ^ε we have

$$-\frac{1}{\varepsilon} f(\phi^\varepsilon)_x = -\left[\frac{1}{\varepsilon} f(\phi) - pf'(\phi) + \varepsilon p^2 \int_0^1 (1-s) f''(\phi - s\varepsilon p)\, ds\right]_x$$

$$\frac{1}{\varepsilon^2}(\beta(\psi^\varepsilon) - \beta(\phi^\varepsilon)) = \frac{2p}{\varepsilon}\beta'(\phi) + p^2 \int_0^1 (1-s)(\beta''(\phi + s\varepsilon p) - \beta''(\phi - s\varepsilon p))\, ds.$$

Using this in the above inequality, after regrouping the terms in their powers of ε, the terms with the factor $\frac{1}{\varepsilon}$ cancel, and we get

$$0 \leqslant \iint \operatorname{sgn}(u^\varepsilon - \phi^\varepsilon)\left[-\phi_t + \left[(f'(\phi))^2 \frac{\phi_x}{2\beta'(\phi)}\right]_x + p^2 A^\varepsilon + \varepsilon(p_t - (p^2 C^\varepsilon)_x)\right] + \operatorname{sgn}(v^\varepsilon - \psi^\varepsilon)\left[-\phi_t + \left[(f'(\phi))^2 \frac{\phi_x}{2\beta'(\phi)}\right]_x - p^2 B^\varepsilon - \varepsilon(p_t - (p^2 C^\varepsilon)_x)\right] dx\, dt,$$

where $A^\varepsilon := \int_0^1 (1-s) f''(\phi - s\varepsilon p)\, ds$, $B^\varepsilon := \int_0^1 (1-s) f''(\phi + s\varepsilon p)\, ds$ and $C^\varepsilon := \int_0^1 (1-s)(\beta''(\phi + s\varepsilon p) - \beta''(\phi - s\varepsilon p))\, ds$.

Note that A^ε, B^ε and C^ε are bounded uniformly in ε, and in fact, as $\varepsilon \to 0$, $A^\varepsilon, B^\varepsilon \to \frac{1}{2} f''(\phi)$ and $C^\varepsilon \to 0$.

3. Assuming as before that $\operatorname{sgn}(u^\varepsilon - \phi^\varepsilon) \to \operatorname{sgn}(u - \phi)$, $\operatorname{sgn}(u^\varepsilon - \phi^\varepsilon) \to \operatorname{sgn}(u - \phi)$, we let $\varepsilon \to 0$. The terms with the factor ε converge to 0, the terms $p^2 A^\varepsilon$ and $p^2 B^\varepsilon$ have the same limit, and so at last we get (14).

As in the previous two sections, we fix the problem of letting $\varepsilon \to 0$ inside $\operatorname{sgn}(\cdot)$ by changing the test functions a little: first considering $\phi^\varepsilon = \phi - \varepsilon\psi$, and then $\phi^\varepsilon = \phi + \varepsilon\psi$, for an appropriate ψ. $\square$

We did not have to restrict ourselves to a 2×2 system. In fact we can use very similar computations in order to obtain a rather general parabolic problem (see below).

5. Accretiveness of semilinear operators

In the next section we get a limit for a more general relaxation limit. However we need to know that the limiting operator is accretive. We do this now.

Consider the operator $B : D(B) \subset L^1(0, \infty, L^1(\mathbb{R}^n)) \to L^1(\mathbb{R}^n)$:

$$Bu = u_t - \sum_{i,j=1}^{n} (a_{ij}(u)u_{x_i})_{x_j} - f, \tag{15}$$

with $D(B) = (C^1 \cap L^1)(0, \infty; (C^2 \cap L^1)(\mathbb{R}^n)), f \in L^1(0, \infty, L^1(\mathbb{R}^n))$ and $a_{ij} \in W^{1,\infty}(\mathbb{R})$ (possibly degenerate) *elliptic*, that is,

$$a_{ij}(z)\xi_i\xi_j \geqslant 0 \quad \text{for all } \xi \in \mathbb{R}^n \text{ and } z \in \mathbb{R}.$$

Let us define

$$Au = -\sum_{i,j=1}^{n} (a_{ij}(u)u_{x_i})_{x_j}, \tag{16}$$

with domain $D(A) = (C^2 \cap L^1)(\mathbb{R}^n)$, so that

$$Bu = u_t + Au - f.$$

Theorem 5.1. *The operator A defined in* (16) *satisfies*:

$$\int_{\mathbb{R}^n} \operatorname{sgn}(u - v)(Au - Av)\, dx \geqslant 0.$$

In particular, A is accretive in $L^1(\mathbb{R}^n)$.

Proof. It is sufficient to show that $\lim_{m \to \infty} \int \beta_m(u - v)(Au - Av)\, d\mu \geqslant 0$, where β_m is a nondecreasing function such that $\lim_m \beta_m(x) \to \operatorname{sgn}(x)$ as $m \to \infty$, $\beta'_m(x) = 0$ if $|x| \geqslant 1/m$ and $|\beta'_m(x)| \leqslant 3m$ otherwise. Then

$$\begin{aligned} I_m &:= \int_{\mathbb{R}^n} \beta_m(u - v)(Au - Av)\, dx \\ &= \sum_{i,j=1}^{n} \int_{\mathbb{R}^n} \beta'_m(u - v)(u - v)_{x_j}(a_{ij}(u)u_{x_i} - a_{ij}(v)v_{x_i})\, dx \\ &= \sum_{i,j=1}^{n} \int_{U_m} \beta'_m(u - v)(u - v)_{x_j}(a_{ij}(u)u_{x_i} - a_{ij}(v)v_{x_i})\, dx, \end{aligned}$$

where we define $U_m = \{|u-v| \leqslant 1/m\}$. Since a_{ij} is Lipschitz, we can find $K>0$ such that $|a_{ij}(u) - a_{ij}(v)| \leqslant K|u-v|$. Thus we have

$$\begin{aligned} I_m = & \sum_{i,j=1}^{n} \int_{U_m} \beta'_m(u-v)(u-v)_{x_j} a_{ij}(u)(u-v)_{x_i}\, dx \\ & + \sum_{i,j=1}^{n} \int_{U_m} \beta'_m(u-v)(u-v)_{x_j}(a_{ij}(u) - a_{ij}(v)) v_{x_i}\, dx \\ \geqslant & -n^2 K \int_{U_m} \beta'_m(u-v)|u-v| \cdot |D(u-v)| \cdot |Dv|\, dx, \end{aligned}$$

where we used the ellipticity condition and the fact that $\beta'_m \geqslant 0$. Note $\beta'_m(s)s \leqslant 3$ for all m. Thus, when we let m go to ∞ in the last integral we get

$$\lim_{m\to\infty} \int_{U_m} \beta'_m(u-v)|u-v||D(u-v)||Dv|\, dx \leqslant 3 \int_{\{|u-v|=0\}} |D(u-v)||Dv|\, dx.$$

Since $Du = Dv$ a.e in $\{u = v\}$ this integral is zero and hence the result. $\square$

To show that B is accretive we can restrict its domain to the functions satisfying a given initial condition: $D(B) = (C^1 \cap L^1)(0,\infty; (C^2 \cap L^1)(\mathbb{R}^n)) \cap \{u : u(x,0) = g(x)\}$. Then we have

Theorem 5.2. *The operator B satisfies*

$$\int_{\mathbb{R}^n \times (0,\infty)} \operatorname{sgn}(u-v)(Bu - Bv)\, dx\, dt \geqslant 0.$$

Proof. The previous theorem takes care of the part of the integral with the x derivatives—we just have to integrate in t. For the time derivative we have, with β_m and U_m as above,

$$\begin{aligned} \iint \beta_m(u-v)(u_t - v_t)\, dx\, dt &= -\iint \beta'_m(u-v)(u-v)_t(u-v)\, dx\, dt \\ &= -\iint_{U_m} \beta'_m(u-v)(u-v)_t(u-v)\, dx\, dt. \end{aligned}$$

As above, we get

$$\limsup_{m} \left| \iint \beta_m(u-v)(u_t - v_t)\, dx\, dt \right| \leqslant C \iint_{\{u=v\}} |u_t - v_t|\ dx\, dt = 0. \qquad \square$$

For the next lemma define $[f,g]_{\mathrm{o}} = \int \operatorname{sgn}(f) g\, dx$ for $f, g \in L^1(\mathbb{R}^n)$.

Lemma 5.3. *If u is a dissipative solution of $Bu = 0$, then*

$$[u - \phi - k, B(\phi + k)]_{\mathrm{o}} + \int_\Omega |u(x,0) - \phi(x,0)|\, dx \geqslant 0, \tag{17}$$

for all $k \in \mathbb{R}$ and $\phi \in D(B)$.

Proof. Given $\phi \in D(B)$, take $\zeta \in C_c^\infty([0,\infty) \times \mathbb{R}^n)$ such that $0 \leqslant \zeta \leqslant 1$, $\zeta(x,t) = 0$ if $|x|^2 + t^2 > 4$, $\zeta(x,t) = 1$ if $|x|^2 + t^2 < 1$. Define $\zeta_m(x,t) = \zeta(x/m^{n+1}, t/m^{n+1})$ and $\phi_m = \phi + k\zeta_m$. Then we have

$$\begin{aligned}
&[u - \phi_m, -B\phi_m]_{\mathrm{o}} \\
&= \iint \operatorname{sgn}(u - \phi_m)\left(\phi_{m,t} - \sum_{i,j=1}^n (a_{ij}(\phi_m)\phi_{m,x_i} + f)_{x_j}\right) dx\, dt \\
&= \iint_{\{|x|^2 + t^2 < m^2\}} \operatorname{sgn}(u - \phi - k)\left(\phi_t - \sum_{i,j=1}^n (a_{ij}(\phi + k)\phi_{x_i})_{x_j} + f\right) dx\, dt \\
&\quad + \iint_{\{|x|^2 + t^2 > m^2\}} \operatorname{sgn}(u - \phi_m)\left(\phi_{m,t} - \sum_{i,j=1}^n (a_{ij}(\phi_m)\phi_{m,x_i})_{x_j} + f\right) dx\, dt \\
&= T_1^m + T_2^m.
\end{aligned}$$

Clearly, when we let $m \to \infty$, $T_1^m \to [u - \phi - k, -B(\phi + k)]_{\mathrm{o}}$. On the other hand $\zeta_{m,x_i}(x,t) = \frac{1}{m^{n+1}}\zeta_{x_i}(x/m^{n+1}, t/m^{n+1})$, $\zeta_{m,t}(x,t) = \frac{1}{m^n}\zeta_t(x/m^{n+1}, t/m^{n+1})$. This vanishes outside $B(0, 2m)$ and is bounded in absolute value by Mm^{-n-1} for some constant M. Denoting by A_m the annulus $\{4m^2 > |x|^2 + t^2 > m^2\}$, we have

$$\begin{aligned}
|T_2^m| &\leqslant \iint_{A_m} |\phi_t| + \frac{|k|M}{m^{n+1}} + \sum_{i,j=1}^n |a'_{ij}(\phi_m)\phi_{x_i}\phi_{m,x_j} + a_{ij}(\phi_m)\phi_{x_i x_j}| + |f|\, dx\, dt \\
&\leqslant \iint_{A_m} |\phi_t| + \frac{|k|M}{m^{n+1}} + C\left(|D\phi|\left(|D\phi| + \frac{|k|M}{m^n}\right) + |D^2\phi|\right) + |f|\, dx\, dt
\end{aligned}$$

where $C > 0$ is a constant depending only on the coefficients a_{ij}

$$\leqslant \hat{C}\frac{|k|M}{m} + \iint_{\{|x|^2 + t^2 > m^2\}} (|\phi| + |f|)\, dx\, dt,$$

with $\hat{C}$ another constant independent of m. Since ϕ and f are summable, it is clear that $T_2^m \to 0$. On the other hand, as in the proof of Theorem 5.1, we have

$$\begin{aligned}
[u - \phi_m, Bu - B\phi_m]_{\mathrm{o}} &\leqslant \int_0^\infty \int_{\mathbb{R}^n} \operatorname{sgn}(u - \phi_m)(u - \phi_m)_t\, dx\, dt, \\
&\leqslant \iint \operatorname{sgn}(u - \phi_m)(u - \phi)\, dx\, dt + \left| \iint_{A_m} \frac{|k|M}{m^{n+1}}\, dx\, dt \right|.
\end{aligned}$$

Now we can let $m \to \infty$, and we recover (18) as in the proof of the same theorem. $\square$

6. Relaxation to a semilinear parabolic equation

Let us now take the following system:

$$\begin{cases} w_t^\varepsilon + \frac{1}{\varepsilon}\sum_{i=1}^n \sum_{j=1}^n f_{ij}(w^\varepsilon)_{x_j} = -\frac{1}{\varepsilon^2}\sum_{i=1}^n (w^\varepsilon - z_i^\varepsilon) \\ z_{i,t}^\varepsilon - \frac{1}{\varepsilon}\sum_{j=1}^n f_{ij}(z_i^\varepsilon)_{x_j} = \frac{1}{\varepsilon^2}(w^\varepsilon - z_i^\varepsilon), \quad i = 1, \dots, n. \end{cases} \tag{18}$$

This is a system where once again the quantities w and z_i are competing. In this case the interaction is linear and the conservation law is nonlinear. Note that this is not a special case of (6), not only because of the nonlinearity, but also due to the dependence on more variables of the transport PDE for z_i. Here we assume that $f_{ij}' \geqslant 0$.

Let us assume for now that this system is a contraction, and that w^ε, z_i^ε have limit u in a suitable norm. An argument similar to the above shows that u should satisfy

$$u_t - \frac{1}{n+1}\sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n (f_{ik}'(u) f_{ij}'(u) u_{x_k})_{x_j} = 0,$$

which we can write as

$$u_t - \frac{1}{n+1}\sum_{j=1}^n \sum_{k=1}^n (a_{jk}(u) u_{x_k})_{x_j} = 0, \tag{19}$$

where $a_{jk}(z) = \sum_{i=1}^n f_{ik}'(z) f_{ij}'(z)$ is a nonnegative definite (possibly degenerate) symmetric matrix: $((a_{jk})) = \sigma^T \sigma$, with $\sigma = ((f_{ij}'))$.

First we show that (18) is in fact a contraction in L^1.

Lemma 6.1. *Let* $(w, \mathbf{z})$ *and* $(\hat{w}, \hat{\mathbf{z}})$ *solve* (18). *Then for any* $0 \leqslant t_1 \leqslant t_2$ *we have*

$$||(w, \mathbf{z})(\cdot, t_1) - (\hat{w}, \hat{\mathbf{z}})(\cdot, t_1)|| \leqslant ||(w, \mathbf{z})(\cdot, t_2) - (\hat{w}, \hat{\mathbf{z}})(\cdot, t_2)||_{L^1(\mathbb{R}_x^n)}. \tag{20}$$

Proof. Suppose $(w, \mathbf{z})$ and $(\hat{w}, \hat{\mathbf{z}})$ are as above. Fix $\delta > 0$ and let $\rho(x)$ be a nonnegative smooth function such that $\rho(x) = |x|$ for $|x| > \delta$, $\rho'(x) > 0$ for $x > 0$ and $\rho'(x) < 0$ for $x < 0$. Then we have

$$\begin{aligned} \frac{d}{dt} R(t) &:= \frac{d}{dt}\int_{\mathbb{R}^n} \rho(w - \hat{w}) + \sum_{i=1}^n \rho(z_i - \hat{z}_i)\, dx \\ &= \int_{\mathbb{R}^n} \rho'(w - \hat{w})(w - \hat{w})_t + \sum_{i=1}^n \rho'(z_i - \hat{z}_i)(z_i - \hat{z}_i)_t\, dx \end{aligned}$$

$$= -\frac{1}{\varepsilon}\sum_{i,j=1}^{n}\int_{\mathbb{R}^n}\rho'(w-\hat{w})\left[(f_{ij}(w)-f_{ij}(\hat{w}))_{x_j}+\frac{1}{n\varepsilon}(w-z_i-\hat{w}+\hat{z}_i)\right]dx$$
$$+\frac{1}{\varepsilon}\sum_{i,j=1}^{n}\int_{\mathbb{R}^n}\rho'(z_i-\hat{z}_i)\left[(f_{ij}(z_i)-f_{ij}(\hat{z}_i))_{x_j}+\frac{1}{n\varepsilon}(w-z_i-\hat{w}+\hat{z}_i)\right]dx.$$

Since $f'_{ij}\geqslant 0$, $\rho'(w-\hat{w})=\rho'(f_{ij}(w)-f_{ij}(\hat{w}))$, except perhaps on the set where $f_{ij}(w)=f_{ij}(\hat{w})$. However, $(f_{ij}(w)-f_{ij}(\hat{w}))_{x_j}=0$ almost everywhere on this set. Therefore, the first term on the first of the above integrals is zero: $\int_{\mathbb{R}^n}[\rho(f_{ij}(w)-f_{ij}(\hat{w}))]_{x_j}\,dx=0$. Similarly, way we conclude that the first term of second integral also vanishes. Thus we have

$$\frac{d}{dt}R(t)=\frac{1}{\varepsilon^2}\sum_{i=1}^{n}\int_{\mathbb{R}^n}[\rho'(z_i-\hat{z}_i)-\rho'(w-\hat{w})][(w-\hat{w})-(z_i-\hat{z}_i)]\,dx.$$

If $w-\hat{w}$ and $z_i-\hat{z}_i$ have the same sign, the above integrand is zero. If either $w\geqslant\hat{w}$ and $z_i\leqslant\hat{z}_i$ or $w\leqslant\hat{w}$ and $z_i\geqslant\hat{z}_i$, then the integrand is $-2(|w-\hat{w}|+|z_i-\hat{z}_i|)\leqslant 0$. Therefore, $R(t_1)\leqslant R(t_2)$, for $0\leqslant t_1\leqslant t_2$. Letting $\delta\to 0$, we get (20). □

We can now check that u is a dissipative solution of (19), that is,

Theorem 6.2. *There exist subsequences w^ε, z_i^ε converging to a dissipative solution of (19), u.*

Proof. 1. From the previous lemma we get the convergence for appropriate subsequences as before. We need to show that for all smooth ϕ,

$$0\leqslant\iint\operatorname{sgn}(u-\phi)\left(-\phi_t+\frac{1}{n+1}\sum_{j,k=1}^{n}(a_{jk}(\phi)\phi_{x_k})_{x_j}\right)dx\,dt. \tag{21}$$

2. As before, a strong solution of (18) is also a dissipative solution, which we will test against $(\phi^\varepsilon,\psi_1^\varepsilon,\dots,\psi_n^\varepsilon)$. Take $\phi^\varepsilon\equiv\phi$ and $\psi_i^\varepsilon=\phi+\varepsilon\psi_{i*}$, where $\psi_{i*}=\sum_{j=1}^{n}f_{ij}(\phi)_{x_j}$. From (9), we get

$$f_{ij}(\psi_i^\varepsilon)=f_{ij}(\phi)+\varepsilon\psi_{i*}f'_{ij}(\phi)+\varepsilon^2\psi_{i*}^2\int_0^1(1-s)f''_{ij}(\phi+\varepsilon s\psi_{i*})\,ds.$$

From the definition of dissipative solution for the system, and (18), we deduce that

$$0\leqslant\iint\operatorname{sgn}(w^\varepsilon-\phi)\left[-\phi_t-\frac{1}{\varepsilon}\sum_{i=1}^{n}\sum_{j=1}^{n}f_{ij}(\phi)_{x_j}-\frac{1}{\varepsilon^2}\sum_{i=1}^{n}(\phi-\psi_i^\varepsilon)\right]$$
$$+\sum_{i=1}^{n}\operatorname{sgn}(z_i^\varepsilon-\psi_i^\varepsilon)\left[-\psi_{i,t}^\varepsilon+\frac{1}{\varepsilon}\sum_{j=1}^{n}f_{ij}(\psi_i^\varepsilon)_{x_j}-\frac{1}{\varepsilon^2}(\psi_i^\varepsilon-\phi)\right]dx\,dt$$

$$\leqslant \iint \operatorname{sgn}(w^\varepsilon - \phi)[-\phi_t] + \sum_{i=1}^{n} \operatorname{sgn}(z_i^\varepsilon - \psi_i^\varepsilon)\left[-\psi_{i,t}^\varepsilon + \sum_{j=1}^{n} ((\psi_{i*} f'_{ij}(\phi))_{x_j} + \varepsilon(\psi_{i*}^2 B_{ij}^\varepsilon)_{x_j})\right] dx\, dt,$$

where

$$B_{ij}^\varepsilon = \int_0^1 (1-s) f''_{ij}(\phi + \varepsilon s \psi_{i*})\, ds.$$

3. The terms with $1/\varepsilon$ all cancel since $\psi_{i*} = \sum_{j=1}^n f(\phi)_{x_j}$. With the same assumptions as before on the convergence of the two $\operatorname{sgn}(\cdot)$ functions in the above integrals, we let $\varepsilon \to 0$. Since $B_{ij}^\varepsilon \to \frac{1}{2} f''_{ij}(\phi)$, when we let $\varepsilon \to 0$ the resulting terms cancel because they have opposite signs. The terms with ε in front vanish in the limit. The rest of the terms add up and we recover (21).

We can conclude the argument by fixing the perturbations as in Section 4, choosing $\phi^\varepsilon = \phi \pm \varepsilon\psi$.

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